Quantum theory of particles and fields as an extension of a probabilistic variational approach

to classical mechanics and classical field theory: I

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Abstract
A theoretical scheme, based on a probabilistic generalization of Hamilton's principle, is elaborated to obtain a unified description of more general dynamical behaviors determined both from a Lagrangian function and by mechanisms not contemplated by this function. Within this scheme, quantum mechanics, classical field theory and a quantum theory for scalar fields are discussed. As a by-product of the probabilistic scheme for classical field theory, the equations of the De Donder–Weyl theory for multi-dimensional variational problems are recovered.

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1. Introduction

The main purpose of this paper is to explore the possibility of reconsidering the problem of the relativistic quantum theory of fields from a different point of view. We will focus on standard fields with no reference to strings.

Let us start from the following general remarks. Suppose we have a classical isolated physical system described by a set of real spacetime functions (fields) $q^i(x)$ $(i = 1, 2, \ldots, n)$, $x = (x^0, x^1, x^2, x^3) \in M$, the spacetime manifold, with the metric tensor $g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$. If a Lagrangian (density)

$$\mathcal{L} = \mathcal{L}(q^i(x), \partial_\mu q^i(x)) \qquad \left(\partial_\mu = \frac{\partial}{\partial x^\mu}, \mu = 0, 1, 2, 3\right) \tag{1}$$

is associated with the system then, as is well known, the field equations determining the functions $q^i(x)$ can be obtained through a variational principle, that is, by requiring that the action

$$A_c = \int_V \mathcal{L}(q^i(x), \partial_\mu q^i(x)) \, \mathrm{d}x \qquad (\mathrm{d}x = \mathrm{d}x^0 \, \mathrm{d}x^1 \, \mathrm{d}x^2 \, \mathrm{d}x^3) \tag{2}$$

be stationary under variations of $q^i(x)$ vanishing at the boundary of V, a domain of M. Then one obtains the partial differential (Euler–Lagrange) equations

$$\partial_\mu \left(\frac{\mathcal{L}}{\partial(\partial_\mu q^i(x))} \right) - \frac{\partial \mathcal{L}}{\partial q^i(x)} = 0 \qquad (i = 1, 2, \ldots, n). \tag{3}$$

In the case of a classical mechanical system with a finite number of degrees of freedom (MSF), a useful and relevant tool for the study of its dynamical properties is given by the Hamiltonian approach. There is essentially one Hamiltonian formulation associated with a given Lagrangian. This formulation has a fundamental role in the transition from classical to quantum theory if this transition is performed on the basis of the standard canonical quantization rules (SCQR). But it is also very relevant and effective in conjunction with the Hamilton–Jacobi theory and general aspects of the one-dimensional variational problem involved in the case of an MSF [1–3].

On the other hand, in the case of continuous systems described by fields, we do not have a univocal Hamiltonian formulation associated with a given classical Lagrangian. The multi-dimensional variational problem (2) is tackled essentially according to two different points of view: one developed mainly in physics and the other in mathematics.

In physics, a continuous system is interpreted as a mechanical system with an infinite number of degrees of freedom (MSI) [4, 5]. The spatial coordinates (x^1, x^2, x^3) have the role of labels of these infinite degrees of freedom. The Hamiltonian formalism is developed on the basis of this mechanical transcription. Such a formulation has a crucial role in the transition to quantum theory of a continuous system, since it allows us to apply the SCQR of quantum mechanics. However, it has a drawback in the fundamental distinction between time and space variables. As is well known, this problem is bypassed in quantum theory by working in the Heisenberg or interaction picture or in the framework of the Feynman's functional approach. However, due to the original point of view, further problems appear, such as the occurrence of infinite quantities. In some cases these problems are bypassed through additional prescriptions, such as renormalization. In these cases, the resulting (effective) quantum field theory has been very successful. It has reached the stage of a paradigm, although accompanied by doubts about its completeness. It runs into difficulties in the case of gravitational fields.

At a classical level, a drawback of the mechanical transcription of a continuous system is that it leads to a Hamilton–Jacobi equation formally of functional type, which frustrates its use. On the other hand, the mathematical analysis of multi-dimensional variational problems does not make essential use of the interpretation of a classical continuous system as an MSI [1, 2, 6]. In this analysis, the space coordinates are treated on the same footing as time: we have four parameters which together take the place that the time alone has in a classical MSF. As a matter of fact there are several approaches to multi-dimensional variational problems [1, 2, 6] which share this point of view. However one relevant approach has been formulated, called the De Donder–Weyl theory [1, 2, 6, 7], which introduces and makes use of a 'Hamiltion' function. In this theory a covariant Hamiltonian formalism is elaborated [8] and a generalized Hamilton–Jacobi equation is obtained, which is not a functional, but a partial differential equation.

At the classical level, the De Donder–Weyl formalism appears more appropriate than the Hamiltonian formalism based on the mechanical transcription. However, if we consider the transition to quantum theory, it is in conflict with the SCQR. In fact these rules require only one canonical momentum (density) associated with each field $q^i(x)$, that is $\partial \mathcal{L}/\partial_0 q^i(x)$ (as happens in the theory based on the mechanical analogy), while in the De Donder–Weyl theory we have four momenta

$$\pi_i^\mu(x) = \frac{\partial \mathcal{L}}{\partial_\mu q^i(x)} \qquad (\mu = 0, 1, 2, 3; i = 1, 2, \ldots, n). \tag{4}$$

Several attempts have been made in order to overcome the above difficulty and to obtain a covariant canonical quantization [9–11, and references therein]. Unfortunately, in these attempts, there is some arbitrariness and some insubstantiality in the probabilistic interpretation.

It seems that the mechanical point of view is necessary, if one insists on the SCQR as the unique tool for a quantum theory of fields. In the past there have been objections to this point of view [12, 13]. As a matter of fact, Born [13] tried to avoid the SCQR and to introduce an alternative quantization procedure based on a result of the multi-dimensional calculus of variations (the Hilbert independence theorem) [14]. But his attempt was not successful.

In this paper, we again make an attempt like that of Born, but firstly we elaborate an appropriate alternative theoretical scheme for the analysis of the transition from classical theory to quantum theory. Developing an approach proposed in a previous work [15], we adopt from the start a probabilistic point of view that is already at the classical level. In this setting we make systematic use of the variational approach, by introducing a generalization of Hamilton's principle. In this way we obtain a scheme more flexible and more directly open to possible extension.

The preliminary aspects of this scheme are discussed in section 2, in the context of classical mechanics. The transition to quantum mechanics is analyzed in section 3; this transition appears more as a natural extension than a conceptual jump. A particular role is covered by time-reversal invariance and by the local or global character of the dynamical equations. We will deduce the request for a mechanical system to be natural and discuss a mechanism for the emergence of complex amplitudes. The case of a discrete random variable (spin) is also analyzed. The probabilistic point of view is applied to classical fields in section 4. We avoid the mechanical transcription, since it leads to an ill-defined probabilistic scheme. However we show that there exists a well-defined probabilistic scheme, in which the usual deterministic classical field theory can be embedded. It is very interesting that, in this scheme, there appear, as a by-product, the equations of the De Donder–Weyl theory. We see that the two Hamiltonian formulations, which are 'equivalent' from the deterministic point of view (in the sense that they give the same equations (3)), have different implications if we adopt a probabilistic point of view.

The transition to a quantum theory of fields is discussed in section 5. This transition, in which the SCQR are avoided, is an extension of the results of sections 3 and 4.

2. Probabilistic variational approach to classical mechanics

We consider a conservative mechanical system with n degrees of freedom and configuration space $\mathbb{R}^n$, described by a Lagrangian L, i.e. a given smooth function of $2n$ independent variables

$$L = L(q, w) \qquad (q = (q^1, q^2, \ldots, q^n), w = (w^1, w^2, \ldots, w^n)). \tag{5}$$

The coordinates q give the configurations of the system. Let us fix a time $\tau > 0$ and make in (5) the substitutions

$$q \to \xi(t), \qquad w \to \dot{\xi}(t) \qquad \left(\dot{} = \frac{\mathrm{d}}{\mathrm{d}t}\right). \tag{6}$$

Then we can consider the action functional

$$I = \int_0^\tau L(\xi(t), \dot{\xi}(t)) \, \mathrm{d}t. \tag{7}$$

According to Hamilton's principle, the temporal evolution of our system is described by its trajectories $q(t)$, which are functions of t such that

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} I(\varepsilon)\bigg|_{\epsilon=0} = \frac{\mathrm{d}}{\mathrm{d}\varepsilon} \int_0^\tau L(q(t) + \varepsilon\eta(t), \dot{q}(t) + \varepsilon\dot{\eta}(t)) \,\mathrm{d}t\bigg|_{\epsilon=0} = 0 \qquad (8)$$

for every smooth function $\eta(t)$ having support in $[0, \tau]$. As is well known, we obtain in this way the Lagrange equations for $q(t)$

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial w^i}(q(t), \qquad \dot{q}(t)) - \frac{\partial L}{\partial q^i}(q(t), \qquad \dot{q}(t)) = 0 \qquad (i = 1, 2, \ldots, n). \qquad (9)$$

Now, we adopt from the start a probabilistic point of view. At each instant $t \in [0, \tau]$, we consider the configuration q a random variable described by a probability density $\rho(q, t)$. We limit ourselves to this assumption only, that is we do not require *a priori* that the temporal evolution is necessarily described by a stochastic process. We will take into account the obvious fact that the total probability

$$\int_{\mathbb{R}^n} \rho(q, t) \,\mathrm{d}q \qquad (\mathrm{d}q = \mathrm{d}q^1 \cdots \mathrm{d}q^n)$$

does not depend on the time t, assuming that this conservation law can be expressed in a local form

$$\partial_t \rho(q, t) + \partial_k j^k(q, t) = 0, \qquad \left(\partial_k = \frac{\partial}{\partial q^k}\right) \qquad (10)$$

through a probability current density $j(q, t) = (j^1(q, t), \ldots, j^n(q, t))$.

We will consider $j(q, t)$ as a further dynamical variable, which, together with $\rho(q, t)$, characterizes the 'state' of the system. As a matter of fact, (10) takes the place or generalizes the link between $q(t)$ and $\dot{q}(t)$ of the standard theory based on the notion of trajectory. We plan to determine (adding initial conditions) $\rho(q, t)$ and $j(q, t)$ directly from the given Lagrangian (5), through an appropriate generalization of Hamilton's principle [15]. To this end, suppose that $\rho(q, t)$ and $j(q, t)$ describe an actual temporal evolution of our system and let us consider the family of subsets $\Sigma = U \times [t_0, t_1]$, where U is a compact domain of $\mathbb{R}^n$ and $0 \leqslant t_0 < t_1 \leqslant \tau$, such that $\rho(q, t) > 0$ for $q, t \in \Sigma$. In each Σ let us make the substitution

$$w \to \frac{j(q, t)}{\rho(q, t)} \qquad (11)$$

in the given Lagrangian (5). Due to the meaning of $j(q, t)$ and $\rho(q, t)$, this substitution is a generalization of (6). Then let us consider the action functional

$$I_\Sigma[j, \rho, \lambda] = \int_\Sigma \mathrm{d}t \,\mathrm{d}q \left[L\left(q, \frac{j(q, t)}{\rho(q, t)}\right) \rho(q, t) + \lambda(q, t)(\partial_t \rho(q, t) + \partial_k j^k(q, t)) \right], \qquad (12)$$

where $\lambda(q, t)$ is a Lagrange multiplier field defined in Σ.

We can now formulate a generalized Hamilton's principle, which replaces (8), as follows:

'The actual functions $\rho(q, t)$ and $j(q, t)$ are such that equation (10) holds in $\mathbb{R}^n \times [0, \tau]$, and in each Σ there exists a function $\lambda(q, t)$ such that

$$\frac{\partial}{\partial \varepsilon_i} I_\Sigma[j + \varepsilon_1 j_0, \rho + \varepsilon_2 \rho_0, \lambda]\bigg|_{\substack{\varepsilon_1=0 \\ \varepsilon_2=0}} = 0 \qquad (i = 1, 2) \qquad (13)$$

for every smooth function $j_0(q, t)$ for which $j_0(q, t) = 0$ for $q \in \partial U$, and every smooth function $\rho_0(q, t)$ such that $\rho_0(q, t_0) = \rho_0(q, t_1) = 0$ and $\rho_0(q, t) = 0$ for $q \in \partial U$.'

By considering $\frac{\partial I_\Sigma}{\partial \varepsilon_1}$, we deduce from this principle

$$\partial_k \lambda(q, t) = \frac{\partial L}{\partial w^k}\left(q, \frac{j(q, t)}{\rho(q, t)}\right), \qquad (k = 1, 2, \ldots, n), \qquad (14)$$

while from $\frac{\partial I_\Sigma}{\partial \varepsilon_2} = 0$ we obtain

$$\partial_t \lambda(q,t) + \frac{j^k(q,t)}{\rho(q,t)} \frac{\partial L}{\partial w^k}\left(q, \frac{j(q,t)}{\rho(q,t)}\right) - L = 0, \tag{15}$$

in each Σ. We note that, in the derivation of (14) and (15) the condition $\rho(q,t) = 0$ for $q \in \partial U$ has no role. However this condition has some implications which will be discussed in the following.

We can solve (14) with respect to j/ρ and substitute the result in (10) and (15). To this end, we assume that our Lagrangian satisfies the condition

$$\det\left(\frac{\partial^2 L}{\partial w^i \partial w^j}(q,w)\right) \neq 0. \tag{16}$$

We see that the variation of ρ in the generalized action principle leads us directly to consider the classical Legendre transformation. If we introduce the Hamiltonian H

$$H(q,p) = w^k p_k - L(q,w), \qquad (p = (p_1, p_2, \ldots, p_n)) \tag{17}$$

where $w = w(q,p)$ is the solution of the system of equations

$$p_i = \frac{\partial L}{\partial w^i}(q,w), \qquad (i = 1, 2, \ldots, n) \tag{18}$$

we obtain, from (14) and (17),

$$\frac{j^k}{\rho} = \frac{\partial H}{\partial p_k}(q, \partial \lambda) \qquad (\partial \lambda = \partial_1 \lambda, \ldots, \partial_n \lambda). \tag{19}$$

Then (15) becomes

$$\partial_t \lambda + H(q, \partial \lambda) = 0 \qquad (q, t \in \Sigma). \tag{20}$$

In the following, we will call *global* any property, equation, function and so on, which is valid or defined on the whole $\mathbb{R}^n \times [0, \tau]$, more precisely for every $q, t, \in \mathbb{R}^n \times [0, \tau]$. Otherwise we will make use of the term *local*.

In the evaluation of the global dynamical variables $\rho(q,t)$ and $j(q,t)$, in which we are interested, we are faced with the field variables $\lambda(q,t)$, which, in principle, can be local. This is related to the presence or absence of a set of points where $\rho(q,t) = 0$. But, *a priori*, we have no knowledge of this set (which we will call *N*). Furthermore, the fields $\lambda(q,t)$ have to satisfy, in any case, some consistency conditions, since they are related, through (19), to the variables ρ and j. However in equation (20) we do not have any terms depending on ρ (and then on the condition $\rho(q,t) > 0$). Therefore, as a result of this decoupling of $\lambda(q,t)$ from ρ, the lack of knowledge of *N* has no influence. The consistency conditions can be solved automatically by promoting the $\lambda(q,t)$ to a global function $S(q,t)$, thus satisfying the global equation

$$\partial_t S(q,t) + H\big(q, \partial S(q,t)\big) = 0 \qquad (q, t \in \mathbb{R}^n \times [0, \tau]). \tag{21}$$

From (19) we deduce that j/ρ can be promoted to a global variable determined by $S(q,t)$. Then we can write for ρ the global equation

$$\partial_t \rho(q,t) + \partial_k\left(\rho(q,t) \frac{\partial H}{\partial p_k}(q, \partial S(q,t))\right) = 0 \qquad (q, t \in \mathbb{R}^n \times [0, \tau]). \tag{22}$$

Equation (21) is the well-known Hamilton–Jacobi equation [1–4]. Its decoupling from (22) is responsible for the basic aspects of classical mechanics. In fact, equation (22) describes a deterministic process. It admits, as particular solutions, probability densities $\rho(q,t)$ with the structure

$$\rho(q,t) = \delta(q - q(t)), \tag{23}$$

where $q(t)$ satisfies the system of first-order differential equations

$$\dot{q}^k(t) = \frac{\partial H}{\partial p_k}(q(t), \partial S(q(t), t)) \qquad (k = 1, 2, \ldots, n). \tag{24}$$

For this $q(t)$, one can deduce, from (21) and (24), the canonical Hamilton's equations and the Lagrange equations (9). We see that the type of randomness determined by a pure Lagrangian approach is essentially related to initial conditions in the configuration space.

From the start we have deliberately avoided references to trajectories or to ensembles of trajectories, with the aim of having a scheme more suitable for possible extensions. If we make use of trajectories in the probabilistic approach, as happens if we assume initially that $j(q, t) = \rho(q, t)v(q, t))$ $(v \in \mathbb{R}^n)$, with $v(q, t)$ independent of ρ and regular in $\mathbb{R}^n \times [0, \tau]$ (a relation which is deduced in the previous scheme), then equations (21) and (22) can also be obtained through alternative well-known classical procedures [16–18]. In this respect we also note that (14) and (15), with j/ρ substituted by the so-called *'geodesic fields' (or control fields)* have already been obtained by Carathéodory in his approach to existence problems for the extremals of the classical action (7) [1, 2, 19]. They were called by him the fundamental equations of the calculus of variations [19]. These ingredients of the Carathéodory approach appear in a natural and unified way in the previous probabilistic scheme.

But the essential advantage of the generalized action principle is that it provides a springboard for a suitable more general scheme. The aim is to have a unified description of more general dynamical behaviors involving both deterministic processes given by a Lagrangian function L and processes not contemplated by this function. As a matter of fact, here we will not be concerned with external (random) actions on our system, but with more general internal mechanisms which govern its dynamics.

The possibility that there can be active additional mechanisms is strictly related to the structure of our generalized action principle. In general, we can implement the property that the total probability is independent of time, through a balance equation involving $\partial_t \rho(q, t)$ and several quantities connected with the dynamical processes taking place in our system. It can be expressed in a form like that of a master equation. So we can naturally extend the previous classical scheme by considering balance equations more general than equation (10). Hence, our approach is based, in general, on two ingredients, a Lagrangian function L and a master-like equation, which can involve additional processes that cannot be ascribed to this function.

However, it is useful to take into account a further aspect of the generalized action principle, which allows us to see classical mechanics from a new perspective. It is concerned with the non-uniqueness of the Lagrangian function L.

In the standard variational approach, based on the trajectories, the arbitrariness of L is expressed by the addition of a total time derivative. We have a similar situation in the case of the generalized action principle, which is strictly related to the validity of the conservation law (10). In fact, due to (10), the Lagrangian L and the Lagrangian L' given by

$$L' = L\left(q, \frac{j}{\rho}\right) + \frac{j^k(q, t)}{\rho(q, t)} \partial_k \chi(q, t) + \partial_t \chi(q, t), \tag{25}$$

where $\chi(q, t)$ is an arbitrary smooth function, are equivalent. They give, through the action integral (12), the same equations of motion. Therefore the choice between L and L' is irrelevant. However we have another class of equivalent Lagrangians, involving only our internal dynamical variables $\rho(q, t)$ and $j(q, t)$. It can be easily verified that, due to (10), the Lagrangian $\tilde{L}$ given by

$$\tilde{L} = L\left(q, \frac{j}{\rho}\right) + d(\rho)\frac{j^k(q, t)}{\rho(q, t)} \partial_k \rho(q, t), \tag{26}$$

where $d(\rho)$ is a given function of ρ, is also equivalent to L. This result is strictly related to the condition that the variation of $\rho(q, t)$ must vanish on the boundary of U. This condition appears superfluous at the classical level, if we take into account the structure of the action functional (12). However the equivalence of L and $\tilde{L}$ allows us to interpret the scheme of classical mechanics from a new point of view. In fact, if we consider $\tilde{L}$ as our Lagrangian function, then we can say that, at the classical level, there also operate virtual internal dynamical mechanisms, generically of diffusive type. Their manifestation is forbidden by the particular relationship between ρ and j, given by the balance equation (10).

The fact that (10) makes irrelevant the coupling between j and $\partial\rho$ appearing in (26), is related to a fundamental invariance property in the dynamical behavior of a mechanical system. Let us assume that the Lagrangian L satisfies the condition

$$L(g, w) = L(g, -w), \tag{27}$$

as happens, in general, for the kinetic energy term. Then, as is well known, we have that the law governing the evolution of our mechanical system, given by the classical scheme, is invariant under time reversal. Formally, the system of equations (10) and (20) (with j given by (19)) is invariant under the (time-reversal) transformations

$$
\begin{aligned}
&t \to t' = -t, \\
&q^i \to q'^i = q^i, \\
&\rho(q, t) \to \rho'(q, t') = \rho(q, t), \\
&\lambda(q, t) \to \lambda'(q, t') = -\lambda(q, t), \\
&q, t \in \Sigma \to q, t' \in \Sigma' = U \times [-t_1, -t_0].
\end{aligned}
\tag{28}
$$

Now the term $j^k \partial_k \rho$ in (26) is not invariant under the transformations (28). Its ineffectiveness in the classical scheme can then be ascribed to the action of the time-reversal invariance.

3. Quantum mechanics

We consider now a particular and relevant extension of the classical scheme.

First of all, in the framework of the generalized action principle, we take $\tilde{L}$ as the Lagrangian function of our system, that is we take into account the virtual internal processes which are present at the classical level. However we do not assume that $\partial_k j^k$ gives all of the contribution to the time derivative of $\rho(q, t)$. We can interpret the field $j(q, t)$, which through j/ρ determines the value of L at the point q, as a quantity related in a broad sense to the transport or convective aspects of the dynamics. Together with ρ, $j(q, t)$ characterizes, as before, the 'state' of our mechanical system. But now, in the balance equation, we take into account an additional non-convective contribution to $\partial_t \rho(q, t)$ by considering, besides j, a current density $j_d(q, t)$ having a given explicit dependence on ρ. Here we limit ourselves to a balance equation which can be still expressed in a local form. So we generalize equation (10) to

$$\partial_t \rho(q, t) + \partial_k j^k(q, t) + \partial_k j_d^k(q, t) = 0, \tag{29}$$

with j_d a certain local function of ρ

$$j_d(q, t) = \mathrm{i}(q, \rho(q, t), \partial_l \rho(q, t), \partial_l \partial_m \rho(q, t), \ldots). \tag{30}$$

As a matter of fact, through the current j_d in the balance equation we consider a real additional mechanism in the dynamical evolution, which is, in a broad sense, of diffusive type. However it also makes real the processes contemplated by $\tilde{L}$, which are virtual when $j_d = 0$. Starting

from $\tilde{L}$ and the extended balance equation (29), the action integral in the generalized action principle becomes

$$\tilde{I}_\Sigma[j, \rho, \lambda] = \int_\Sigma \mathrm{d}t \, \mathrm{d}q \left\{ \left[L\left(q, \frac{j}{\rho}\right) + d(\rho)\frac{j^k}{\rho}\partial_k \rho \right] \rho(q, t) \right.$$
$$\left. + \lambda(\partial_t \rho + \partial_k j^k + \partial_k i^k(q, \rho, \partial_l \rho, \ldots)) \right\}. \tag{31}$$

From (31) we deduce the generalization of (14) and (15), that is

$$\partial_k \lambda - d(\rho)\partial_k \rho = \frac{\partial L}{\partial w^k}\left(q, \frac{j}{\rho}\right) \qquad (k = 1, 2, \ldots, n), \tag{32}$$

$$\partial_t \lambda + \frac{j^k}{\rho}\frac{\partial L}{\partial w^k}\left(q, \frac{j}{\rho}\right) - L + d(\rho)\partial_k j^k + (\partial_k \lambda)\frac{\partial i^k}{\partial \rho}$$
$$- \partial_k\left((\partial_l \lambda)\frac{\partial i^l}{\partial(\partial_k \rho)}\right) + \partial_k \partial_l\left((\partial_m \lambda)\frac{\partial i^m}{\partial(\partial_k \partial_l \rho)}\right) - \cdots = 0, \tag{33}$$

in every Σ.

By introducing the Hamiltonian H, we obtain the system of the two coupled equations

$$\partial_t \lambda + H(q, \partial\lambda - d(\rho)\partial\rho) + d(\rho)\partial_k\left(\rho\frac{\partial H}{\partial p_k}(q, \partial\lambda - d(\rho)\partial\rho)\right)$$
$$+ (\partial_k \lambda)\frac{\partial i^k}{\partial \rho} - \partial_k\left((\partial_l \lambda)\frac{\partial i^l}{\partial(\partial_k \rho)}\right) + \partial_k \partial_l\left((\partial_m \lambda)\frac{\partial i^m}{\partial(\partial_k \partial_l \rho)}\right) - \cdots = 0 \tag{34}$$

$$\partial_t \rho + \partial_k\left(\rho\frac{\partial H}{\partial p_k}(q, \partial\lambda - d(\rho)\partial\rho)\right) + \partial_k i^k = 0, \tag{35}$$

in every Σ.

In general, in the system of equations (34) and (35) we lose time reversal invariance. However we impose on our extension of the classical scheme the explicit condition that the invariance under the transformation (28) is preserved, notwithstanding the presence of the current $i(q, \rho, \partial_l \rho, \ldots)$ and of $d(\rho)\partial\rho$. As a matter of fact we have two types of diffusive processes, which can have a competitive or antagonistic role. As we will see, the requirement of time-reversal invariance selects an important class of mechanical systems. Moreover, it allows us to deduce completely, from the knowledge of $\tilde{L}$, the specific structure of j_d.

A condition for the invariance under the transformations (28) is that $d(\rho)$ must be necessarily different from zero when $j_d \neq 0$, as can be easily seen from equations (35). Moreover, by considering (34), we deduce from (28) that this invariance requires

$$\frac{1}{2}[H(q, \partial\lambda - d(\rho)\partial\rho) - H(q, -\partial\lambda - d(\rho)\partial\rho)]$$
$$+ d(\rho)\partial_i\left[\frac{\rho}{2}\left(\frac{\partial H}{\partial p_i}(q, \partial\lambda - d(\rho)\partial\rho) - \frac{\partial H}{\partial p_i}(q, -\partial\lambda - d(\rho)\partial\rho)\right)\right]$$
$$+ (\partial_k \lambda)\frac{\partial i^k}{\partial \rho} - \partial_k\left((\partial_l \lambda)\frac{\partial i^l}{\partial \partial_k \rho}\right) + \partial_k \partial_l\left((\partial_m \lambda)\frac{\partial i^m}{\partial(\partial_k \partial_l \rho)}\right) - \cdots = 0. \tag{36}$$

From the structure of equation (36) and the requirement that it must be satisfied identically by λ and ρ, we deduce that $H(q, \partial\lambda - d(\rho)\partial\rho) - H(q, -\partial\lambda - d(\rho)\partial\rho)$ must be linear in $\partial\lambda$, or, in other words, that $H(q, p)$ must be quadratic in p. So, the request that the system (34) and (35) be invariant under the time-reversal transformations (28), forces us to limit ourselves to

natural [20] Lagrangian systems, that is to mechanical systems described by a Lagrangian of the type

$$L = L(q, w) = \tfrac{1}{2} m_{ij}(q) w^i w^j - V(q), \tag{37}$$

where $m_{ij}(q)(= m_{ji}(q))$ is a positive-definite mass matrix.

From (36) we also deduce

$$\frac{\partial i^m}{\partial(\partial_k \partial_l \rho)} = \frac{\partial i^m}{\partial(\partial_k \partial_l \partial_j \rho)} = \cdots = 0,$$

so that

$$j_d = i(q, \rho, \partial\rho). \tag{38}$$

Hereafter we will fix our attention on the Lagrangian (37). We have

$$H(q, p) = \tfrac{1}{2} m^{ij}(q) p_i p_j + V(q), \tag{39}$$

where m^{ij} is the inverse matrix of m_{ij}. Condition (36) of time-reversal invariance becomes

$$\partial_k \lambda \left[\rho d(\rho) \partial_j m^{ij} + \frac{\partial i^k}{\partial \rho} - \partial_j \left(\frac{\partial i^k}{\partial(\partial_j \rho)} \right) \right] + \partial_k \partial_j \lambda \left[\rho d(\rho) m^{kj} - \frac{\partial i^k}{\partial(\partial_j \rho)} \right] = 0. \tag{40}$$

In order that (40) be satisfied identically by λ and ρ, we must have a relationship between $d(\rho)$ and the current $i(q, \rho, \partial\rho)$. It is given by

$$i^k(q, \rho, \partial\rho) = m^{kj}(q) \rho d(\rho) \partial_j \rho + \tilde{i}^k(q) \qquad (k = 1, 2, \ldots, n). \tag{41}$$

It can be easily verified that (41) also makes equation (35) time-reversal invariant, provided $\partial_k \tilde{i}^k(q) = 0$. So $\tilde{i}^k(q)$ is irrelevant and can be completely neglected. We conclude that i^k must be a diffusion current of the type

$$i^k = -D^{kj}(q, \rho) \partial_j \rho, \tag{42}$$

with a matrix diffusion coefficient determined by the mass matrix and by the 'coupling' function $d(\rho)$ (so that, in general, it depends on ρ), i.e.

$$D^{kj}(q, \rho) = -\rho d(\rho) m^{kj}(q). \tag{43}$$

So, we have worked out a formulation of a class of dynamical behaviors, which generalizes the classical scheme, but preserves in any case the property of time-reversal invariance. This class is characterized by the action functional (31), the Lagrangian (37) and the relationship (41) (with $\tilde{i}^k = 0$). The law governing these dynamical behaviors follows from (34) and (35). We have

$$\partial_t \lambda + \tfrac{1}{2} m^{ij} \partial_i \lambda \partial_j \lambda + V + \tfrac{1}{2} m^{ij} (\rho d^2(\rho))' \partial_i \rho \partial_j \rho - \partial_i (m^{ij} \rho d^2(\rho) \partial_j \rho) = 0, \tag{44}$$

$$\partial_t \rho + \partial_i (\rho \, m^{ij} \partial_j \lambda) = 0, \tag{45}$$

in every Σ where $\rho > 0 ((\,)' = \tfrac{d}{d\rho}(\,))$.

It is useful to note that the system (44) and (45) has a Hamiltonian structure, with $\rho(q, t)$ and $\lambda(q, t)$ playing the role of conjugate fields variables. From (31) and (32) it follows easily that (44) and (45) can be deduced directly from the standard variational principle applied to the action functional

$$A_\Sigma[\lambda, \rho] = \int_\Sigma dt \, dq [-\rho \partial_t \lambda - \mathcal{H}_e], \tag{46}$$

with the effective Hamiltonian density $\mathcal{H}_e$ given by

$$\mathcal{H}_e(\rho, \partial\rho, \lambda, \partial\lambda) = +\rho \left[\tfrac{1}{2} m^{kl} \partial_k \lambda \partial_l \lambda + V \right] + \tfrac{1}{2} \rho d^2(\rho) m^{kl} \partial_k \rho \partial_l \rho. \tag{47}$$

(The variation of λ must vanish on the boundary of Σ.) A particular case of (47) has been previously postulated in the Fényes theory [21].

Now, our problem is to determine the global field variables $\rho(q, t)$ and $j(q, t)$, having at our disposal the two coupled local equations (44) and (45) and the local relationship (32). From this we deduce, as before, that the local (in principle) fields $\lambda(q, t)$ must satisfy some consistency conditions. In this respect we observe that if $\rho d^2(\rho)$ is a smooth function of ρ, regular at $\rho = 0$, there appears no effective restriction on the set of points in $\mathbb{R}^n \times [0, \tau]$ in which equations (44) and (45) can be considered valid. In this case, the Hamiltonian $\mathcal{H}_e$ is a smooth function in every domain of $\mathbb{R}^n \times [0, \tau]$. As a consequence, in this case, $\lambda(q, t)$ can be promoted to a global field variable $S(q, t)$, as happens in the classical scheme. But now the equation for $S(q, t)$ also involves $\rho(q, t)$. Such a situation will be called the extended classical domain. We can make a further step and go beyond the extended classical domain, by considering the situation in which $\rho d^2(\rho)$ is singular at $\rho = 0$. In this case the above straightforward globalization does not apply. However we can ask if this singularity can be absorbed in a canonical transformation of the field variables $\rho(q, t)$ and $\lambda(q, t)$, allowing, in terms of the new variables, a straightforward global description of dynamical behaviors not contemplated by the extended classical domain.

By fixing attention on a set Σ, let us express ρ and λ in terms of two new variables

$$\rho = P(u, v) \qquad \lambda = \Lambda(u, v). \tag{48}$$

We will assume that the transformation defined in (48) is canonical, i.e. it has the property that there exists a function $F(u, v)$ such that

$$P d\Lambda = u du + dF. \tag{49}$$

By considering u and v as fields, $u = u(q, t)$ and $v = v(q, t)$, the system (44) and (45) will be transformed in a new system involving $u = u(q, t)$ and $v = v(q, t)$, still having a canonical structure.

A necessary and sufficient condition that (48) be canonical is

$$\frac{\partial(P, \Lambda)}{\partial(u, v)} = \frac{\partial P}{\partial u} \frac{\partial \Lambda}{\partial v} - \frac{\partial P}{\partial v} \frac{\partial \Lambda}{\partial u} = +1. \tag{50}$$

Now, suppose that $\rho d^2(\rho)$ is singular at $\rho = 0$. Since we consider ρ, j and j_d, which appear in the global balance equation (29), as global field variables, we deduce from (41) that we must have in any case that $\rho d(\rho)$ is regular at $\rho = 0$. This means that $\rho d^2(\rho)$ must have a pole of first order at $\rho = 0$. So we can write

$$\rho d^2(\rho) = \frac{(a/2)^2}{\rho} + g(\rho) \qquad (a > 0), \tag{51}$$

where a is a constant and $g(\rho)$ is a smooth function of ρ, regular at $\rho = 0$.

By considering the canonical transformation (48), we have

$$- \rho \partial_t \lambda - \mathcal{H}_e = -u(q, t) \partial_t v(q, t) - \mathcal{H}'_e(u, \partial u, v, \partial v) + \partial_t F(u, v), \tag{52}$$

where according to (51), the transformed effective Hamiltonian density is given by

$$\mathcal{H}'_e = \frac{1}{2} m^{kl} \left\{ \left[P \left(\frac{\partial \Lambda}{\partial u} \right)^2 + \frac{(a/2)^2}{P} \left(\frac{\partial P}{\partial u} \right)^2 \right] \partial_k u \partial_l u - \left[P \left(\frac{\partial \Lambda}{\partial v} \right)^2 + \frac{(a/2)^2}{P} \left(\frac{\partial P}{\partial v} \right)^2 \right] \partial_k v \partial_l v \right.$$
$$\left. - 2 \left[P \left(\frac{\partial \Lambda}{\partial u} \right) \left(\frac{\partial \Lambda}{\partial v} \right) + \frac{(a/2)^2}{P} \left(\frac{\partial P}{\partial u} \right) \left(\frac{\partial P}{\partial v} \right) \right] \partial_k u \partial_l v \right\}$$
$$+ \frac{1}{2} m^{kl} g(P) \partial_k P \partial_l P + P V. \tag{53}$$

Our aim is to eliminate, through an appropriate choice of the canonical transformation, the singular dependence of $\mathcal{H}'_e$ on P, at $P = 0$. If this happens, the field variables involved

in this transformation can be promoted straight-forwardly to global variables. To this end, on the basis of the structure of the coefficients of the quadratic form appearing in (53) and of the condition (50), we consider a transformation such that

$$\frac{a}{2}\frac{\partial P}{\partial u} = P\frac{\partial \Lambda}{\partial v} \qquad \frac{a}{2}\frac{\partial P}{\partial v} = -P\frac{\partial \Lambda}{\partial u}. \tag{54}$$

If (50) and (54) are satisfied, we have

$$P\left(\frac{\partial \Lambda}{\partial u}\right)\left(\frac{\partial \Lambda}{\partial v}\right) + \frac{(a/2)^2}{P}\left(\frac{\partial P}{\partial u}\right)\left(\frac{\partial P}{\partial v}\right) = 0,$$

$$P\left(\frac{\partial \Lambda}{\partial u}\right)^2 + \frac{(a/2)^2}{P}\left(\frac{\partial P}{\partial u}\right)^2 = P\left(\frac{\partial \Lambda}{\partial v}\right)^2 + \frac{(a/2)^2}{P}\left(\frac{\partial P}{\partial v}\right)^2$$

$$= \frac{a}{2}. \tag{55}$$

So the singular dependence on P, at $P = 0$, disappears. Now, let us determine the above canonical transformation. From (54) it follows that $\log P$ and $\Lambda/(a/2)$ must be the real and the imaginary part of an analytic function $f(z)$

$$\log P + \mathrm{i}\frac{\Lambda}{a/2} = f(z), \tag{56}$$

with $z = u + \mathrm{i}v$. Furthermore, due to (50), we have

$$|f'(z)|^2 = \left(\frac{\partial \log P}{\partial u}\right)^2 + \left(\frac{\partial}{\partial u}\frac{\Lambda}{a/2}\right)^2 = \frac{1}{P \cdot a/2}. \tag{57}$$

Then, by making use of (56), we deduce that $f(z)$ satisfies the equation

$$\left|\frac{\mathrm{d}}{\mathrm{d}z}\,\mathrm{e}^{f(z)/2}\right|^2 = \frac{1}{2a}. \tag{58}$$

Therefore

$$\mathrm{e}^{\frac{f(z)}{2}} = \left(\frac{1}{2a}\right)^{\frac{1}{2}}\mathrm{e}^{\mathrm{i}\varphi}(z + c_1 + \mathrm{i}c_2), \tag{59}$$

where φ, c_1, c_2 are real constants.

In conclusion, the canonical transformation which satisfies (54) is given by

$$P(u, v) = \frac{(u + c_1)^2 + (v + c_2)^2}{2a}$$

$$\frac{\Lambda(u, v)}{a} = \arg(u + c_1 + \mathrm{i}(v + c_2)) + \varphi. \tag{60}$$

(In Σ we can select a determination of arg.) So, as a consequence of (51), in this case the probability density is related to the squared modulus of a complex amplitude, while Λ is connected with the phase of this amplitude. Such a situation will be called, in general, the extended quantum domain. The constants c_1 and c_2 can be absorbed in a further canonical transformation: $u' = u + c_1$, $v' = v + c_2$. Furthermore, it is useful to use the scaled field variables

$$\psi_1 = \frac{u'}{(2a)^{\frac{1}{2}}} \qquad \psi_2 = \frac{v'}{(2a)^{\frac{1}{2}}},$$

so that (60) becomes

$$P(\psi_1, \psi_2) = \psi_1^2 + \psi_2^2, \qquad \frac{\Lambda(\psi_1, \psi_2)}{a} = \arctan\frac{\psi_1}{\psi_2} + \varphi. \tag{61}$$

By considering the effective Lagrangian

$$\mathcal{L}_e = -\rho \partial_t \lambda - \mathcal{H}_e, \tag{62}$$

which gives the system (45), we deduce, from (52)–(55) and (61), that $\mathcal{L}_e$ is also given by

$$\mathcal{L}_e = -a(\psi_1 \partial_t \psi_2 - \psi_2 \partial_t \psi_1) + \frac{1}{2} a^2 m^{kl}(\partial_k \psi_1 \partial_l \psi_1 + \partial_k \psi_2 \partial_l \psi_2) - V(q)\big(\psi_1^2 + \psi_2^2\big)$$
$$- 2g\big(\psi_1^2 + \psi_2^2\big) m^{kl}(\psi_1 \partial_k \psi_1 + \psi_2 \partial_k \psi_2) \cdot (\psi_1 \partial_l \psi_1 + \psi_2 \partial_l \psi_2), \tag{63}$$

where we have neglected a term involving a total time derivative.

Although we have derived (63) in a set Σ, we see that all the coefficients of the quadratic form involving $\partial_k \psi_1$, $\partial_l \psi_2$ are regular functions, even at $\rho = \psi_1^2 + \psi_2^2 = 0$. So there is no effective restriction to considering $\mathcal{L}_e$, as given by (63), on the whole $\mathbb{R}^n \times [0, \tau]$ and to promoting straight-forwardly $\psi_1(q, t)$ and $\psi_2(q, t)$ to global field variables. From (63) we can deduce a system of two equations for ψ_1 and ψ_2, which globalizes the system (44) and (45).

In this respect, it is useful to consider a further canonical transformation, by introducing a complex field ψ and its conjugate

$$\psi = \psi_1 + \mathrm{i}\psi_2 \qquad \psi^* = \psi_1 - \mathrm{i}\psi_2. \tag{64}$$

In terms of the field variables ψ and ψ^*, the Lagrangian (63) becomes

$$\mathcal{L}_e = +\mathrm{i}\frac{a}{2}(\psi^* \partial_t \psi - \psi \partial_t \psi^*) - \frac{1}{2} a^2 m^{kl} \partial_k \psi^* \partial_l \psi - V \psi^* \psi$$
$$- \frac{g(\psi^*\psi)}{2} m^{kl}(\psi^* \partial_k \psi + \psi \partial_k \psi^*)(\psi^* \partial_l \psi + \psi \partial_l \psi^*). \tag{65}$$

From (65) we deduce a system of two equations for ψ and ψ^*. One of these equations is the complex conjugate of the other. In general, there is a coupling between the two equations. The usefulness of the canonical fields ψ and ψ^* stems from the fact that they allow us to determine a single and relevant circumstance in which we obtain two decoupled equations. This happens when

$$g(\psi^*\psi) = 0. \tag{66}$$

In this case, we have a situation analogous (but more symmetrical) to that of the classical domain, in which a canonical field is decoupled from the other. In the classical domain the decoupling is responsible for the existence of deterministic trajectories. The decoupling obtained when (66) is satisfied, is responsible for the linearity of the resulting equation for ψ (or ψ^*). In this case we speak of the quantum domain. When (66) is satisfied, we deduce from (65)

$$\mathrm{i}a\partial_t \psi(q, t) = -\frac{a^2}{2} \partial_i(m^{ij}(q)\partial_j \psi(q, t)) + V(q)\psi(q, t). \tag{67}$$

For

$$a = \hbar, \tag{68}$$

(67) is the Schrödinger equation which governs the quantum mechanics of a natural mechanical system. If $g(\psi^*\psi) \neq 0$, we obtain a nonlinear Schrödinger equation.

It is useful to stress that the need for the field variables like $\psi_1(q, t)$ and $\psi_2(q, t)$ is related to the fact that our problem is to determine the global field variables $\rho(q, t)$ and $j(q, t)$, while at an intermediate step we have to introduce a local (in principle) field variable, that is $\lambda(q, t)$. We are then faced with the solution of some matching conditions, which among other things would require knowledge (the set N) not given *a priori*. These conditions are automatically solved by considering new field variables for which there is no obstacle to being

extended globally in straight-forward way. Without any conditions we could not say that the system of local equations (44) and (45) is equivalent, for $g(\psi^*\psi) = 0$, to the Schrödinger equation, which is valid globally. As a matter of fact (44) and (45) is a system of generalized Madelung hydrodynamic equations [15, 18]. The inequivalence between the Schrödinger equation and the Madelung equations has been already discussed by Wallstrom [22]. However it is significant that, when (66) is satisfied, the system of local equations (44) and (45) can be interpreted in terms of classical Brownian trajectories, as shown in Nelson's stochastic mechanics approach [16, 17, 23, 24].

Our previous analysis provides, as a particularly relevant case, a scheme for a quantization procedure. In this respect we note that, in the framework of the canonical quantization rules, we have, in general, a problem with the ordering of the product of noncommuting operators, as happens when an Hamiltonian of the type (39) is considered generally. Our scheme, through equation (67), allows us to give an answer to this problem. Furthermore we remark that the consideration of the Lagrangian $\mathcal{L}_e$ (63) or (65) is also useful from another point of view. By applying to $\mathcal{L}_e$ the Noëther theorem [3], we can select the relevant conserved physical quantities, associated with the symmetry properties of our mechanical system. When (66) is satisfied, these quantities are in correspondence with the expectation values of the operators of the standard quantum theory [18].

From the theoretical point of view, our approach predicts different dynamical behaviors, according to the structure of $d(\rho)$. This structure cannot be given *a priori*, without further considerations. In principle, it must be argued on physical grounds.

The quantization procedure afforded by the previous scheme requires that the mechanical system be natural, a property not necessarily contemplated by the canonical quantization rules. We could also develop a 'quantum' theory of a non-natural mechanical system, on the basis of the action (31) or the system (34) and (35). The resulting 'quantum'-dynamical behavior would break the time reversal invariance of the classical level. We note that a mechanical system with relativistic kinematics is non-natural. In this case, we are also faced with a 'homogeneous' problem in the calculus of variations [2, 19].

In the case of 'homogeneous' problems, or problems in parametric form, condition (16) is not satisfied. In order to treat these cases (singular Lagrangians), within our approach, we need further elaboration as done in the calculus of variations or in the Dirac approach to constrained Hamiltonian systems [29]. The limiting procedure of Rothe [30] could also be useful. These aspects will be investigated elsewhere.

We conclude this section with a provisional discussion of another extension of the classical scheme, which can be formulated in the framework of our approach.

Suppose we consider the time evolution in the interval $[0, \tau]$ of a dynamical system whose configurations are described by a discrete variable assuming a finite number N of values. These values will be labelled by a discrete index running from 1 to N. At each instant $t \in [0, \tau]$ we consider the configuration of the system a discrete random variable described by a set of probabilities $p_\alpha(t)$ ($\alpha = 1, 2, \ldots, N$).

As a first step, in order to take into account of the fact that

$$\sum_{\alpha=1}^{N} p_\alpha(t)$$

does not depend on t, we introduce a quite general balance equation in the form

$$\dot{p}_\alpha(t) + \sum_{\beta=1}^{N} (\gamma_{\alpha\beta}(t) - \gamma_{\beta\alpha}(t)) = 0 \qquad (\alpha = 1, 2, \ldots, N). \tag{69}$$

The non-local and discrete equation (69) takes the place of (10). We plan to determine variationally $p_\alpha(t)$ and $\gamma_{\alpha\beta}(t)$. In this respect, here we will assume that the functions $p_\alpha(t)$, for each α, are such that they have at most a finite number of zeros in the interval $[0, \tau]$. As a second step we introduce a 'Lagrangian' $\mathcal{L}(\xi)$, a smooth function of a real variable ξ, and a real symmetric matrix $U_{\alpha\beta}$, time independent, which will be connected with the extension of the local substitution (11). We will assume that

$$\mathcal{L}(-\xi) = \mathcal{L}(\xi) \tag{70}$$

and

$$U_{\alpha\beta} = U_{\beta\alpha} \neq 0 \qquad \forall\, \alpha, \beta. \tag{71}$$

Suppose now that $p_\alpha(t)$ and $\gamma_{\beta\alpha}$ give an actual temporal evolution of our system and let us consider the family of subset Γ of $[0, \tau]$, formed by finite unions of disjoint closed internals in which $p_\alpha(t) > 0$, for every α. As a final step, in each Γ we extend (11) by the 'non-local' substitution

$$\xi \to \frac{\gamma_{\alpha\beta}(t)}{p_\alpha^{\frac{1}{2}}(t) U_{\alpha\beta} p_\beta^{\frac{1}{2}}(t)}. \tag{72}$$

Then we replace the action functional (12) by

$$I_\Gamma[\gamma_{\alpha\beta}, p_\alpha, \lambda_\alpha] = \int_\Gamma dt \sum_{\alpha=1}^{N} \left\{ \sum_{\beta=1}^{N} \mathcal{L}\left(\frac{\gamma_{\alpha\beta}(t)}{p_\alpha^{\frac{1}{2}}(t) U_{\alpha\beta} p_\beta^{\frac{1}{2}}(t)} \right) p_\alpha^{\frac{1}{2}}(t) U_{\alpha\beta} p_\beta^{\frac{1}{2}}(t) \right.$$
$$\left. - \lambda_\alpha(t) \left(\dot{p}_\alpha(t) + \sum_{\beta=1}^{N} (\gamma_{\alpha\beta}(t) - \gamma_{\beta\alpha}(t)) \right) \right\}. \tag{73}$$

By applying straight-forwardly the generalized action principle we deduce from (73)

$$\lambda_\alpha(t) - \lambda_\beta(t) = \frac{d\mathcal{L}}{d\xi}\left(\frac{\gamma_{\alpha\beta}(t)}{p_\alpha^{\frac{1}{2}}(t) U_{\alpha\beta} p_\beta^{\frac{1}{2}}(t)} \right) \qquad (\alpha, \beta = 1, \ldots, N; t \in \Gamma). \tag{74}$$

By introducing again a Legendre transformation

$$\mathcal{H}(\eta) = \eta\xi - \mathcal{L}(\xi), \tag{75}$$

we can write

$$\gamma_{\alpha\beta}(t) = p_\alpha^{\frac{1}{2}}(t) U_{\alpha\beta} p_\beta^{\frac{1}{2}}(t) \frac{d\mathcal{H}}{d\eta}(\lambda_\alpha(t) - \lambda_\beta(t))$$
$$(\alpha, \beta = 1, \ldots, N; t \in \Gamma). \tag{76}$$

Then we obtain the system of equations:

$$\dot{\lambda}_\alpha - \sum_{\beta=1}^{N} U_{\alpha\beta} \frac{p_\beta^{\frac{1}{2}}}{p_\alpha^{\frac{1}{2}}} \mathcal{H}(\lambda_\alpha - \lambda_\beta) = 0 \tag{77}$$

$$\dot{p}_\alpha + 2 \sum_{\beta=1}^{N} p_\alpha^{\frac{1}{2}}(t) U_{\alpha\beta} p_\beta^{\frac{1}{2}}(t) \frac{d\mathcal{H}}{d\eta}(\lambda_\alpha - \lambda_\beta) = 0 \tag{78}$$
$$(\alpha, \beta = 1, \ldots, N; t \in \Gamma),$$

where we have made use of (70), (71) and (76).

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The system (77) and (78) defines an Hamiltonian structure. It can be deduced directly from the action functional

$$A_\Gamma[\lambda_\alpha, p_\alpha] = \int_\Gamma \left[-\sum_{\alpha=1}^N p_\alpha(t)\dot{\lambda}_\alpha(t) + \sum_{\alpha,\beta=1}^N p_\alpha^{\frac{1}{2}}(t) U_{\alpha\beta} p_\beta^{\frac{1}{2}}(t) \mathcal{H}(\lambda_\alpha - \lambda_\beta) \right]. \tag{79}$$

We see that the local character of the system (77) and (78) is determined by the non-analytic square root dependence on $p_\alpha(t)$ of the action functional (79). Then we can obtain an explicit description of a global dynamical behavior, if we can absorb this dependence in a canonical transformation involving variables which can be extended globally without any condition. However, in order to carry out this plan, we have to select appropriately the structure of $\mathcal{H}(\eta)$. As a matter of fact our goal can be realized quite easily, by choosing as our 'Hamiltonian'

$$\mathcal{H}(\eta) = b \cos\frac{\eta}{a} \qquad (a > 0), \tag{80}$$

where a and b are real constants. We have then immediately the appropriate canonical transformation, which, written in the standard form involving complex quantities, is given by

$$\psi_\alpha(t) = p_\alpha^{\frac{1}{2}} e^{i\frac{\lambda_\alpha(t)}{a}}. \tag{81}$$

The action functional (79) becomes

$$A_\Gamma[\lambda_\alpha, p_\alpha] = A_\Gamma[\psi_\alpha] = \int_\Gamma \left[i\frac{a}{2} \sum_{\alpha=1}^N \left(\psi_\alpha^* \dot{\psi}_\alpha - \psi_\alpha \dot{\psi}_\alpha \right) + \sum_{\alpha,\beta=1}^N \psi_\alpha^* b U_{\alpha\beta} \psi_\beta \right]. \tag{82}$$

In terms of the variables $\psi_\alpha(t)$ we obtain straight-forwardly a global description of the dynamical behavior. As a matter of fact $\psi_\alpha(t)$ which are solutions of the evolution equations are analytic functions of Γ, so that our assumption on the zeros of $p_\alpha(t)$ is satisfied.

We consider two generalizations of the previous result. We can introduce a possible constant shift in the stationary point of $\gamma_{\alpha\beta}(t)$, by adding in the integrand of equation (73) a term linear in $\gamma_{\alpha\beta}(t)$

$$-\sum_{\alpha,\beta=1}^N \theta_{\alpha\beta} \gamma_{\alpha\beta}(t), \tag{83}$$

where the given real matrix $\theta_{\alpha\beta}$ is antisymmetric ($\theta_{\alpha\beta} = -\theta_{\beta\alpha}$). On the other hand, through an appropriate limiting procedure, we can also consider the case in which some elements of the matrix $U_{\alpha\beta}$ are equal to zero. Then we obtain the final global effective action functional

$$A_\Gamma[\psi_\alpha] = \int_0^\tau \left[i\frac{a}{2} \sum_{\alpha=1}^N \left(\psi_\alpha^* \dot{\psi}_\alpha - \psi_\alpha \dot{\psi}_\alpha^* \right) - \sum_{\alpha,\beta=1}^N \psi_\alpha^* h_{\alpha\beta} \psi_\beta \right], \tag{84}$$

with a generic Hermitian matrix $h_{\alpha\beta}$ given by

$$h_{\alpha\beta} = -b U_{\alpha\beta} e^{-i\frac{\theta_{\alpha\beta}}{a}}. \tag{85}$$

From (84) we deduce the standard discrete Schrödinger equation

$$i\hbar \dot{\psi}_\alpha(t) = \sum_{\beta=1}^N h_{\alpha\beta} \psi_\beta(t) \qquad (\alpha = 1, 2, \ldots, N), \tag{86}$$

which governs the quantum dynamics of a discrete random variable, like the spin.

We finally note that it is possible to consider also the situation in which we have both continuous and discrete random variables.

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4. Probabilistic variational approach to classical field theory

Let us come back to our physical system described classically by a set of real fields $q^i(x)$ and the Lagrangian density (1). By following the theoretical scheme of section 2, we adopt from the start a probabilistic point of view and, at the same time, we plan to generalize the standard variational principle based on the action (2).

At first sight, the immersion of the deterministic classical field theory in a probabilistic scheme could be realized immediately if a field is considered as an MSI. As a matter of fact the mechanical transcription allows us to obtain a Hamilton–Jacobi equation. In this case, we have a functional equation, involving a functional $S([q(\vec{x}), t])$, for each t, of the configurations $q(\vec{x})$ of our system of fields $(\vec{x} = (x^1, x^2, x^3))$. However, we meet with difficulties if we want to place at the side of $S([q(\vec{x}), t])$, a functional probability density $\rho([q(\vec{x}), t])$. There is no Lebesgue measure in an infinite-dimensional space and there are problems with a functional divergence theorem. So we are faced with a probabilistic scheme which is already ill-defined at the classical level, before the transition to quantum theory. Furthermore, space and time are treated in an asymmetric way. On the other hand, as will be discussed in the following, it is possible to embed deterministic classical field theory in a simple and well-defined probabilistic scheme in which the above difficulties are avoided.

At each spacetime point x, we treat the values $q = (q^1, q^2, \ldots, q^n)$ of the set of our fields as random variables described by a probability density $\rho(q, x)$. We have now that the total probability

$$\int_{\mathbb{R}^n} \rho(q, x) \, dq$$

does not depend on the spacetime point x. We assume again that this independence can be expressed in a local form

$$\partial_\mu \rho(q, x) + \partial_k j_\mu^k(q, x) = 0 \qquad (\mu = 0, 1, 2, 3). \tag{87}$$

So, together with $\rho(q, x)$, we have now four probability current densities $j_\mu(q, x) = \left(j_\mu^1(q, x), \ldots, j_\mu^n(q, x)\right)$ through which the 'state' of the system is characterized. In terms of $\rho(q, x)$ and $j_\mu(q, x)$ we can extend in a natural way the generalized Hamilton's principle of section 2 to fields.

It is formally convenient to take into account the property that the Lagrangian density (1) is a function of q and of $4n$ independent variables $w_\mu = (w_\mu^1, \ldots, w_\mu^n)$,

$$\mathcal{L} = \mathcal{L}(q, w_\mu). \tag{88}$$

Now, suppose that $\rho(q, x)$ and $j_\mu(q, x)$ describe an actual evolution of our system in the spacetime. We consider the family of subsets $\Omega_+ = U \times Z$, where U and Z are compact domains of $\mathbb{R}^n$ and M, respectively, such that $\rho(q, x) > 0$ for $q, x \in \Omega_+$. In each Ω_+, we make the substitution (analogous to (11)

$$w_\mu \rightarrow \frac{j_\mu(q, x)}{\rho(q, x)}$$

in the given Lagrangian (88) and then we consider the action functional

$$I_{\Omega_+}[j_\mu, \rho, \lambda^\sigma] = \int_{\Omega_+} dx \, dq \left[\mathcal{L}\left(q, \frac{j_\mu(q, x)}{\rho(q, x)}\right) \rho(q, x) + \lambda^\nu \left(\partial_\nu \rho(q, x) + \partial_k j_\nu^k(q, x)\right) \right], \tag{89}$$

where, in order to take into account equations (87), we have introduced four Lagrange multiplier fields $\lambda^\mu(q, x)$, defined in Ω_+. Now we demand that the actual $\rho(q, x)$ and $j_\mu(q, x)$ are such that $I_{\Omega_+}[j_\mu, \rho, \lambda^\sigma]$ is stationary under smooth independent variations $\delta\rho(q, x)$ of

$\rho(q, x)$ and $\delta j_\mu(q, x)$ of $j_\mu(q, x)$, with $\delta\rho(q, x) = 0$ for $q, x \in \partial U \times \partial Z$ and $\delta j_\mu(q, x) = 0$ for $q \in \partial U$. Furthermore equations (87) must hold.

By considering the variation of $j_\mu(q, x)$ we obtain from (89)

$$\partial_k \lambda^\nu(q, x) = \frac{\partial \mathcal{L}}{\partial w_\nu^k}\left(q, \frac{j_\mu(q, x)}{\rho(q, x)}\right) \qquad (k = 1, 2, \ldots, n; \nu = 0, 1, 2, 3). \tag{90}$$

The variation of ρ gives

$$\partial_\nu \lambda^\nu(q, x) + \frac{j_\mu(q, x)}{\rho(q, x)}\frac{\partial \mathcal{L}}{\partial w_\nu^k}\left(q, \frac{j_\mu(q, x)}{\rho(q, x)}\right) - \mathcal{L}\left(q, \frac{j_\mu(q, x)}{\rho(q, x)}\right) = 0. \tag{91}$$

Now let us assume that

$$\det\left[\frac{\partial^2 \mathcal{L}}{\partial w_\nu^i \partial w_\sigma^j}(q, w_\mu)\right] \neq 0. \tag{92}$$

Then we can solve (90) with respect to j_μ/ρ. Through this solution, equations (87) and (91) can be expressed in an appropriate and useful form, by again making use of the Legendre transformation.

Let us introduce a generalized 'Hamiltonian' function $\mathcal{H}$

$$\mathcal{H} = \mathcal{H}(q, \pi^\mu) = w_\nu^k \pi_k^\nu - \mathcal{L}(q, w_\mu) \qquad \left(\pi^\mu = \left(\pi_1^\mu, \ldots, \pi_n^\mu\right)\right), \tag{93}$$

where $w_\nu^k(q, \pi^\mu)$ is the solution of the system of equations

$$\pi_i^\nu = \frac{\partial \mathcal{L}}{\partial w_\nu^i}(q, w_\mu). \tag{94}$$

Then we have

$$\frac{j_\nu^k}{\rho} = \frac{\partial \mathcal{H}}{\partial \pi_k^\nu}(q, \partial\lambda^\mu) \qquad (\partial\lambda^\mu = (\partial_1\lambda^\mu, \ldots, \partial_n\lambda^\mu)). \tag{95}$$

So we obtain the system of equations

$$\partial_\nu \lambda^\nu(q, x) + \mathcal{H}(q, \partial\lambda^\sigma(q, x)) = 0, \tag{96}$$

$$\partial_\mu \rho(q, x) + \partial_k\left(\rho\frac{\partial\mathcal{H}}{\partial\pi_k^\mu}(q, \partial\lambda^\sigma(q, x))\right) = 0 \qquad (q, x \in \Omega_+). \tag{97}$$

Due to the absence of an explicit restriction connected with the set of points in $\mathbb{R}^n \times M$ where $\rho(q, x) = 0$, we can reconsider (96) and (97) as equations in $\mathbb{R}^n \times M$, free from constraints. They are the version, in the framework of the previous probabilistic scheme for classical field theory, of equations (21) and (22) for an MSF. Through (96) we have obtained a generalization to field theory of the Hamilton–Jacobi equation. However this generalization is again a partial differential equation, and not a functional equation, as happens in the standard canonical field theory. Furthermore the space coordinates are treated on the same footing as the time. These four parameters together take the place that the time alone had in previous sections. The system (96) and (97) satisfies manifestly relativistic invariance, if $\mathcal{L}$, as we will always assume, is a Lorentz scalar. The generalized 'Hamiltonian' $\mathcal{H}$, given by (93), is not an energy density, but a Lorentz scalar. Another feature of our approach to field theory is the occurrence of a system of four equations for ρ, in place of only the one equation (22) for an MSF. As a consequence (96) is now not decoupled from (97). The functions $\lambda^\mu(q, x)$, besides satisfying (96), have to guarantee the consistency of the system (97).

The Euler–Lagrange equations of the standard deterministic classical field theory can be considered as a particular consequence of the system (96) and (97).

Let $q(x) = (q^1(x), \ldots, q^n(x)))$ be a classical field. It can be easily verified that (97) admits as a particular solution the probability density

$$\rho(q, x) = \delta(q - q(x)) \tag{98}$$

if, and only if, the field $q(x)$ is such that

$$\partial_\sigma q^k(x) = \frac{\partial \mathcal{H}}{\partial \pi_k^\sigma}(q(x), \partial \lambda^\mu(q(x), x)). \tag{99}$$

On the other hand, by deriving (96) with respect to q^i, we see that $q(x)$ must also satisfy the equation

$$\partial_\nu \partial_i \lambda^\nu(q(x), x) = -\frac{\partial \mathcal{H}}{\partial q^i}(q(x), \partial \lambda^\mu(q(x), x)). \tag{100}$$

By taking into account the Legendre transformation (93), we deduce from (99)

$$\partial_i \lambda^\mu(q(x), x) = \frac{\partial \mathcal{L}}{\partial w_\mu^i}(q(x), \partial_\nu q(x)) \equiv \pi_i^\mu(x). \tag{101}$$

Then equation (100) gives

$$\partial_\nu \left(\frac{\partial \mathcal{L}}{\partial w_\nu^i}(q(x), \partial_\mu q(x)) \right) - \frac{\partial \mathcal{L}}{\partial q^i}(q(x), \partial_\mu q(x)) = 0,$$

that is the Euler–Lagrange equations (3) for the field $q(x)$.

A function $q(x)$ which is a solution to (99), with $\lambda^\mu(q, x)$ a solution to (96), is said to be embedded in a geodesic field [2, 6, 25]. As we have seen, such a function is an extremal, i.e. a solution to (3). It can be shown that any extremal $q(x)$ can be embedded in a geodesic field, in some neighborhood of the surface $q = q(x)$ [1, 2, 6, 7, 25].

We can take directly as unknown the fields $\partial_i \lambda^\mu(q(x), x) = \pi_i^\mu(x)$. Then we can consider directly the system

$$\partial_\mu q^i(x) = \frac{\partial \mathcal{H}}{\partial \pi_i^\mu}(q(x), \pi^\nu(x)), \tag{102}$$

$$\partial_\mu \pi_i^\mu(x) = -\frac{\partial \mathcal{H}}{\partial q^i}(q(x), \pi^\nu(x)), \tag{103}$$

which is the relativistic covariant version, in classical field theory, of the canonical Hamilton's equations.

As a matter of fact, equations (96), (99), (102), (103), which we have been deduced through our probabilistic approach, have already been obtained on the basis of different considerations, in the deterministic De Donder–Weyl theory developed for multi-dimensional variational problems [1, 2, 6–8, 14, 25]. In the framework of the deterministic classical field theory, they have been also reobtained elsewhere [26, 27].

We note that, in the De Donder–Weyl theory for classical fields, the relevant aspect of manifest relativistic covariance compels to have four 'momentum density' $\pi_i^\mu(x)$ conjugates to each classical field $q^i(x)$. This makes the standard Poisson brackets formalism and then the transition to a quantum theory problematic. However this redundance of conjugate variables $\pi_i^\mu(x)$ can be reduced through proper constraints. As a matter of fact these constraints are contained in equations (102).

We can consider a family of space-like surfaces which covers simply the manifold M. Such a family can be related to a description of the dynamical evolution of our system. However, in the following, for simplicity, we will limit ourselves to surfaces having x_0 constant.

Now the three equations in (102)

$$\partial_m q^i(x) = \frac{\partial \mathcal{H}}{\partial \pi_i^m}(q(x), \pi^0(x), \pi^n(x)) \qquad (m = 1, 2, 3) \tag{104}$$

can be interpreted as constraints for the fields $\pi^n(x)$ $(n = 1, 2, 3)$.

By solving them, we will obtain

$$\pi_i^m(x) = F_i^m(q(x), \pi^0(x), \partial_n q(x)). \tag{105}$$

Then we can introduce the function

$$\mathcal{H}_c(q(x), \pi^0(x), \partial_m q(x)) = \mathcal{H}(q(x), \pi^0(x), F^m(q(x), \pi^0(x), \partial_n q(x)))$$
$$- (\partial_m q^i(x)) F_i^m(q(x), \pi^0(x), \partial_n q(x)). \tag{106}$$

We have

$$F_i^m(q(x), \pi^0(x), \partial_n q(x)) = -\frac{\partial \mathcal{H}_c}{\partial(\partial_m q^i(x))}. \tag{107}$$

The system (102) and (103) is then reduced to

$$\partial_0 q^i(x) = \frac{\partial \mathcal{H}_c}{\partial \pi_i^0} \tag{108}$$

$$\partial_0 \pi_i^0(x) = -\left(\frac{\partial \mathcal{H}_c}{\partial q_i} - \partial_m \frac{\partial \mathcal{H}_c}{\partial(\partial_m q^i(x))}\right), \tag{109}$$

that is the system of equations of the standard Hamiltonian formalism [4], related to the description of a classical field as an MSI. Really, it follows from (93) that $\mathcal{H}_c$ is the standard Hamiltonian density.

Then we see that if we carry out the elimination of the conjugate fields $\pi_i^n(x)$, the resulting theory (which is the standard one) meets with the difficulties mentioned at the beginning of this section. On the other hand, if we keep the complete manifest covariant structure of the De Donder–Weyl theory, then this theory, as we have shown, can be embedded in a simple and well-defined probabilistic scheme, related to a relativistic invariant generalized partial differential equation of Hamilton–Jacobi type. In such a scheme it is possible to contemplate quantum extensions, as in the previous section, which are disengaged from the usual SCQR. This will be done in the following section.

Due to (89) and (95), the basic equations of the previous scheme, that is (96) and (97), can be derived directly from the standard variational principle applied to the action integral

$$A_\Omega[\rho, \lambda^\mu] = \int_\Omega \mathrm{d}x \, \mathrm{d}q [-\lambda^\nu(q, x)\partial_\nu \rho(q, x) + \mathcal{H}(q, \partial \lambda^\mu)\rho(q, x)], \tag{110}$$

where Ω is now a generic domain of $\mathbb{R}^n \times M$. The variations of λ^μ must vanish on the boundary of Ω. From the point of view of standard classical field theory, we see then that only $\lambda^0(q, x)$ has a conjugate variable, that is $\rho(q, x)$, while for $\lambda^m(q, x)$ $(m = 1, 2, 3)$ such conjugate variables are absent. Therefore three of equations (97), that is

$$\partial_m \rho(q, x) + \partial_k \left(\rho \frac{\partial \mathcal{H}_l}{\partial \pi_k^m}(q, \partial \lambda^\sigma(q, x))\right) = 0 \qquad (m = 1, 2, 3) \tag{111}$$

have to be considered as constraints of the dynamical evolution, in analogy with what happens for the system (102). We note that the constraints which appear in the previous approach are intrinsically related to the property that the generalized Hamiltonian is a scalar, and not an energy density. As a matter of fact, due to (92), we are not treating singular Lagrangians. It seems that these constraints have a role different from that given by the Dirac approach.

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We conclude this section with the following remarks. First of all, the previous probabilistic approach to classical field theory requires, in general, some caution. This is related to condition (92), which makes possible a transition from the Lagrangian formalism to a covariant Hamiltonian formalism. In the case of a classical Dirac field, it is well known that the Lagrangian density is linear in the derivatives of the field components so that condition (92) cannot be satisfied. Another problem, of technical character, appears if we have constraints between the components of classical fields, as happens in the case of vector fields. The role and meaning of these cases in our probabilistic approach will be investigated elsewhere. In the following we will limit our considerations to scalar fields having the property of satisfying condition (92).

Since we assumed that our classical Lagrangian density (1) does not depend explicitly on x, we can consider the invariance of our probabilistic theory under spacetime translations, that is under the transformation

$$y = (x, q) \rightarrow y' = (x + \varepsilon, q) \qquad \varepsilon = (\varepsilon^0, \varepsilon^1, \varepsilon^2, \varepsilon^3). \tag{112}$$

Then, from (112) and a straight-forward application of the Noether's theorem to the action (110), we deduce the local conservation law

$$\partial_\mu \left(T_\nu^\mu(q, x) \rho(q, x) \right) + \partial_i Q_\nu^i(q, x) = 0, \tag{113}$$

with

$$T_\nu^\sigma(q, x) = \partial_k \lambda^\sigma(q, x) \frac{\partial \mathcal{H}}{\partial \pi_k^\nu}(q, \partial \lambda^\mu) + g_\nu^\sigma \left(\mathcal{H} - \partial_k \lambda^\alpha \frac{\partial \mathcal{H}}{\partial \pi_k^\alpha} \right). \tag{114}$$

Q_ν^i is a function of q, $\rho(q, x)$, $\partial \lambda^\mu(q, x)$.

So, our approach contemplates a random energy–momentum tensor $T_\nu^\mu(q, x)$ given by (114). If we consider the mean of $T_\nu^\mu(q, x)$ at the point x

$$\overline{T}_\nu^\mu(q, x) = \int_{\mathbb{R}^n} T_\nu^\mu(q, x) \rho(q, x) \, \mathrm{d}q \tag{115}$$

we deduce from (113)

$$\partial_\mu \overline{T}^\mu(x) = 0. \tag{116}$$

In the case of the solution (98), we obtain from (114) and (115)

$$\overline{T}_\nu^\sigma(x) = T_{c\nu}^\sigma(x) = \frac{\partial \mathcal{L}}{\partial w_\sigma^i}(q(x), \partial_\mu q(x)) \partial_\nu q^i(x) - g_\nu^\sigma \mathcal{L}(q(x), \partial_\mu q(x)), \tag{117}$$

i.e. the standard result of the deterministic classical field theory.

5. The quantum extension for scalar fields

We develop a quantum theory for scalar fields, starting from the probabilistic scheme of the previous section and following the approach of section 3. We will limit our considerations to a set of n real fields described by a 'natural' Lagrangian function

$$\mathcal{L}(q, w_\mu) = \tfrac{1}{2} \eta_{ij}(q) w_\nu^i w^{\nu j} - V(q), \tag{118}$$

where $\eta_{ij}(q)$ is positive definite $\forall q$ ($\eta_{ij}(q) = \eta_{ji}(q)$, the inverse matrix is denoted by $\eta^{ij}(q)$). The Hamiltonian function (93) in this case becomes

$$\mathcal{H}(q, \pi^\mu) = \tfrac{1}{2} \eta^{ij}(q) \pi_{\nu i} \pi_j^\nu + V(q). \tag{119}$$

As a first step of an extension of the scheme of the previous section, we take into account, in the balance equation (87), an additional contribution to $\partial_\mu \rho(q, x)$. This is given by current

densities $i_\mu^k(q, \rho(q, x), \partial_i \rho(q, x), \ldots)$ having a local dependence on ρ and on its derivatives with respect to q^i. So equation (87) is extended to

$$\partial_\mu \rho(q, x) + \partial_k j_\mu^k(q, x) + \partial_k i_\mu^k(q, \rho(q, x), \partial_j \rho(q, x), \ldots) = 0. \tag{120}$$

Through i_μ^k we introduce dynamical mechanisms of diffusive type. These mechanisms are internal so that, due to the homogeneity of spacetime, they are assumed to have no explicit dependence on x. However, due to the structure of i_μ^k, we are also faced with the problem of ensuring that, in the theory which we are developing, the isotropy of spacetime is not destroyed. To this end, we note that, as in the case of an MSF, the transition from (87) to (120) makes the relevant choice of a Lagrangian density which, at the classical level, is equivalent to $\mathcal{L}(q, j_\mu(q, x)/\rho(q, x))$. As a matter of fact, when (87) is satisfied, the Lagrangian density

$$\widetilde{\mathcal{L}} = \mathcal{L}\left(q, \frac{j_\mu}{\rho}\right) + D(\rho) a^v \frac{j_v^k(q, x)}{\rho(q, x)} \partial_k \rho(q, x), \tag{121}$$

where $D(\rho)$ is a given function of ρ and a^v is a given constant 4-vector, is equivalent to $\mathcal{L}$.

Then, as a basis for a quantum theory of our system, we consider the action functional

$$\widetilde{I}_{\Omega_+}[j_\mu, \rho, \lambda^\sigma] = \int_{\Omega_+} \mathrm{d}x \, \mathrm{d}q \left\{ \left[\mathcal{L}\left(q, \frac{j_\mu}{\rho}\right) + D(\rho) a^v \frac{j_v^k}{\rho} \partial_k \rho \right] \rho \right.$$
$$\left. + \lambda^v \left(\partial_v \rho + \partial_k j_v^k + \partial_k i_v^k(q, \rho, \partial_l \rho, \ldots) \right) \right\}. \tag{122}$$

Equations (121) and (122) are the generalization to our fields of equations (26) and (31) for mechanical systems.

In the case of non-relativistic mechanical systems we were faced with the breaking of time-reversal invariance, which is related to the choice of an arrow of the time axis. In the case of fields, the introduction of a constant 4-vector a^v, that is a privileged direction in the Minkowski space, breaks the isotropy of the spacetime. At the classical level we have no problem, since the additional term in equation (121) is ineffective. This is not the case, in general, if we start from the action functional (122). However, the isotropy of the spacetime can be still preserved by choosing appropriately the current i_μ^k. As a matter of fact, the request for the spacetime isotropy allows us to determine completely i_μ^k, starting from the knowledge of $\widetilde{\mathcal{L}}$. In fact, on the basis of this request, we deduce from (118), (121) and (122) that

$$i_v^k(q, \rho, \partial_l \rho, \ldots) = a_v \eta^{kj}(q) \rho D(\rho) \partial_j \rho, \tag{123}$$

which is a generalization of (41).

Then we deduce from (118) and (122) the Lorentz invariant system of equations

$$\partial_v \lambda^v + \tfrac{1}{2} \eta^{ij} \partial_i \lambda_v \partial_j \lambda^v + V(q) + a_v a^v \left[\tfrac{1}{2} \eta^{ij} (\rho D^2(\rho))' \partial_i \rho \partial_j \rho - \partial_i (\eta^{ij} \rho D^2(\rho) \partial_j \rho) \right] = 0, \tag{124}$$

$$\partial_\mu \rho + \partial_k (\rho \eta^{kj} \partial_j \lambda_\mu) = 0 \qquad ((\mu = 0, 1, 2, 3), (q, x \in \Omega_+)), \tag{125}$$

which appears as a straight-forward generalization to fields of the system (44) and (45), obtained in the quantum theory of an MSF.

The system (124) and (125) can be derived directly from the action integral

$$A_{\Omega_+}[\rho, \lambda^\mu] = \int_{\Omega_+} \mathrm{d}x \, \mathrm{d}q \left[-\lambda^v(q, x) \partial_v \rho(q, x) + \mathcal{H}(q, \partial \lambda^\mu) \rho + \frac{a_v a^v}{2} \rho D^2(\rho) \eta^{kl} \partial_k \rho \partial_l \rho \right], \tag{126}$$

where $\mathcal{H}$ is given by (119) (the variations of λ^μ must vanish on the boundary of Ω_+).

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The last term in equation (126) has the same structure which we found in the case of an MSF. So we can speak for fields of an extended classical domain or of a quantum domain, according to the behavior of $\rho D^2(\rho)$. We will focus our attention on the quantum domain. This will be characterized, as in the case of mechanical systems (equation 51, $g(\rho) = 0$), by requiring

$$a_\nu a^\nu \rho D^2(\rho) = \frac{(f/2)^2}{\rho} \qquad (f > 0), \tag{127}$$

where f is constant. We have assumed in (127) that $a_\nu a^\nu > 0$, i.e. the currents i_ν^k are timelike, as can be expected on physical grounds.

In the quantum domain (124) becomes

$$\partial_\nu \lambda^\nu + \frac{1}{2} \eta^{ij} \partial_i \lambda_\nu \partial_j \lambda^\nu + V(q) + \frac{f^2}{4} \left(\frac{1}{2} \eta^{ij} \frac{\partial_i \rho \partial_j \rho}{\rho^2} - \frac{1}{\rho} \partial_i (\eta^{ij} \partial_j \rho) \right)$$

$$= \partial_\nu \lambda^\nu + \frac{1}{2} \eta^{ij} \partial_i \lambda_\nu \partial_j \lambda^\nu + V(q) - \frac{f^2}{2} \frac{\partial_i (\eta^{ij} \partial_j \rho^{1/2})}{\rho^{1/2}}$$

$$= 0, \qquad (q, x) \in \Omega_+. \tag{128}$$

The system (125) and (128) has a Lorentz invariant local hydrodynamic form, which generalizes the Madelung hydrodynamic equations related to non-relativistic quantum mechanics. The system is also invariant under the transformations involving spacetime inversion

$$\begin{aligned}
x^\mu &\to x'^\mu = -x^\mu, \\
q^i &\to q'^i = q^i, \\
\rho(q, x^\mu) &\to \rho'(q, x'^\mu) = \rho(q, x^\mu), \\
\lambda^\mu(q, x^\nu) &\to \lambda'^\mu(q, x'^\nu) = -\lambda^\mu(q, x^\nu).
\end{aligned} \tag{129}$$

We note that while, in the case of mechanical systems, the constant a of equation (51), analogous to f, has the dimension of an action (as a matter of fact $a = \hbar$, according to (68), in the case of our fields the constant f has the dimension of a (spatial) density of an action, as a consequence of our previous equations (126) and (127). This means that, by considering $\hbar$, f can be related to a fundamental quantity having the dimension of a spatial length.

We will see some consequences of the system (125) and (128), by assuming that, as in some usual models of field theory (Klein–Gordon, λq^4, . . .),

$$V(q) \geqslant 0, \qquad V(q) \to +\infty \quad \text{for} \quad |q| \to +\infty. \tag{130}$$

Furthermore, we will consider the random energy–momentum tensor $T_{Q_\nu}^\sigma(q, x)$ in the quantum case. This can be obtained by adding to the classical expression (114) the 'quantum potential' which appears in equation (128). Taking into account of (119), we have

$$T_{Q_\nu}^\sigma(q, x) = \eta^{ij}(q) \partial_i \lambda^\sigma \partial_j \lambda_\nu + g_\nu^\sigma \left(-\frac{f^2}{2} \frac{\partial_i (\eta^{ij} \partial_j \rho^{1/2})}{\rho^{1/2}} + V(q) - \frac{1}{2} \eta^{ij} \partial_i \lambda_\mu \partial_j \lambda^\mu \right), \tag{131}$$

for $(q, x) \in \Omega_+$.

In particular, the random energy density $\varepsilon^Q(q, x)$ and the random momentum densities $P_m^Q(q, x)$ ($m = 1, 2, 3$) are given by

$$\varepsilon^Q(q, x) = T_{Q0}^0(q, x) = \frac{1}{2} \eta^{ij} \partial_i \lambda^0 \partial_j \lambda^0 + \frac{1}{2} \eta^{ij} \partial_i \lambda^m \partial_j \lambda^m + V(q) - \frac{f^2}{2} \frac{\partial_i (\eta^{ij} \partial_j \rho^{1/2})}{\rho^{1/2}}, \tag{132}$$

$$P_m^Q(q, x) = T_{Qm}^0(q, x) = \eta^{ik} \partial_k \lambda^0 \partial_i \lambda_m \qquad (m = 1, 2, 3). \tag{133}$$

The simplest solutions of the system (125) and (128) can be obtained by considering the case

$$\partial_i \lambda_\mu(q, x) = 0 \qquad (i = 1, 2, \ldots, n; \mu = 0, 1, 2, 3). \tag{134}$$

In such a case, (125) is solved automatically by $\rho(q, x)$ independent of x, that is $\rho(q, x) = \widetilde{\rho}(q)$. Furthermore (128) (with $\tilde{\rho}$) necessarily requires that $\partial_\nu \lambda^\nu$ is also independent of x, that is, due to (134),

$$\partial_\nu \lambda^\nu(q, x) = -w, \tag{135}$$

where w is a constant. Then (128) becomes

$$-\frac{f^2}{2} \frac{\partial_i(\eta^{ij} \partial_j \widetilde{\rho}^{1/2})}{\widetilde{\rho}^{1/2}} + V(q) = w \tag{136}$$

for q such that $\widetilde{\rho}(q) > 0$.

This equation can be translated in a global form, by setting

$$\widetilde{\rho}(q) = \psi^2(q), \tag{137}$$

so that we are led to replace (136) with

$$\left(-\frac{f^2}{2} \partial_i(\eta^{ij} \partial_j \psi(q))\right) + V(q)\psi(q) = w\psi(q), \tag{138}$$

that is a stationary Schrödinger equation with the constant f, instead of $\hbar$.

Let us denote by w_r the eigenvalues of the 'Hamiltonian' operator

$$\hat{H} = -\frac{f^2}{2} \partial_i(\eta^{ij} \partial_j \cdot) + V(q).$$

We order them according to

$$0 \leqslant w_0 < w_1 < \ldots.$$

We conclude that our system (125) and (128) admits as solutions invariant states, which are associated with the eigenvalues w_r of $\hat{H}$. In an invariant state, corresponding to a particular eigenvalue w_s, the energy–momentum tensor is well defined, constant and given by

$$T_{Q\nu}^{\ \sigma} = g_\nu^\sigma w_s. \tag{139}$$

Due to (139), we see that the existence of several non-degenerate vacuum states, each having a constant finite energy density, is a consequence of our equations. We will call the state corresponding to w_0 the fundamental vacuum, while the other states associated with w_r $(r > 0)$ may be called virtual vacua. The energy gap between the fundamental vacuum and a virtual vacuum is infinite. We note that, in our approach, there is no substantial distinction between the case of interacting fields and the non-interacting one. The fundamental vacuum also exists for interacting fields. In the traditional approach, in the case of interacting fields, the existence of the vacuum state must be, in some way, postulated. In the case of free fields, the traditional approach gives an infinite zero point energy, but this is also infinite in any finite spatial volume, while our w_0 is finite.

The previous aspects of our approach may be relevant in connection with the dark energy problem [28]. We note also that, as a consequence of (130), the eigenfunctions $\psi_r(q)$ associated with the invariant states are rapidly decreasing for $|q| \to +\infty$. Then we have that the fluctuations

$$\overline{(q_i - \overline{q}_i)(q_j - \overline{q}_j)} = \int dq(q_i - \overline{q}_i)(q_j - \overline{q}_j)\psi_0^2(q) \tag{140}$$

($\bar{q}_i = \int dq q_i \psi_0^2(q)$, the $\psi_r(q)$ are supposed normalized) are finite. In the traditional approach, if we consider, for example, the real Klein–Gordon field, the analogous quantity $\langle 0| \hat{q}^2(x) |0\rangle$ ($\hat{q}$ the operator valued Heisenberg field) is infinite.

To further simplify our discussion, in the following we will limit ourselves to the case of only one scalar field, assuming also that $\eta_{ij}(q) = \eta(q) = \eta$ (positive constant).

Now, as next steps, we will examine space-independent solutions and time-independent solutions. The first type corresponds to solutions such that

$$\frac{\partial}{\partial q}\lambda^m(q, x) = 0 \qquad (m = 1, 2, 3) \tag{141}$$

but $\partial\lambda^0/\partial q \neq 0$, in general.

With $\rho(q, x) = \tilde{\rho}(q, x^0)$, as a consequence of (141), the system (125) and (128) becomes

$$\frac{\partial}{\partial x^0}\tilde{\rho}(q, x^0) + \frac{1}{\eta}\frac{\partial}{\partial q}\left(\tilde{\rho}\frac{\partial}{\partial q}\lambda_0(q, x)\right) = 0 \tag{142}$$

$$\frac{\partial}{\partial x^0}\lambda_0(q, x) + \frac{\partial}{\partial x^m}\lambda^m(q, x) + \frac{1}{2\eta}\left(\frac{\partial}{\partial q}\lambda_0(q, x)\right)^2 + V(q) - \frac{f^2}{2\eta}\frac{1}{\tilde{\rho}^{1/2}}\frac{\partial^2\tilde{\rho}^{1/2}}{\partial q^2} = 0. \tag{143}$$

From (142) we deduce $\lambda^0(q, x) = \lambda(q, x^0)$, while (141) and (143) require that $\frac{\partial}{\partial x^m}\lambda^m(q, x) = f(x^0)$. Writing

$$f(x_0) = \frac{\partial}{\partial x^0}\int^{x^0} f(x'^0)\,\mathrm{d}x'^0 \equiv \frac{\partial}{\partial x^0}\Lambda(x^0),$$

we can absorb $\Lambda(x^0)$ within $\lambda(q, x^0)$, by considering $\widetilde{\lambda}(q, x^0) = \lambda(q, x^0) + \Lambda(x^0)$. Then we arrive at the equations

$$\frac{\partial}{\partial x^0}\tilde{\rho}(q, x^0) + \frac{1}{\eta}\frac{\partial}{\partial q}\left(\tilde{\rho}\frac{\partial}{\partial q}\tilde{\lambda}(q, x^0)\right) = 0, \tag{142'}$$

$$\frac{\partial}{\partial x^0}\tilde{\lambda}(q, x^0) + \frac{1}{2\eta}\left(\frac{\partial}{\partial q}\tilde{\lambda}(q, x^0)\right)^2 + V(q) - \frac{f^2}{2\eta}\frac{1}{\tilde{\rho}^{1/2}}\frac{\partial^2\tilde{\rho}^{1/2}}{\partial q^2} = 0. \tag{143'}$$

According to section 3, this system is the local form of the global linear Schrödinger equation

$$\mathrm{i}f\frac{\partial}{\partial x^0}\widetilde{\psi}(q, x^0) = -\frac{f^2}{2\eta}\frac{\partial}{\partial q^2}\widetilde{\psi}(q, x^0) + V(q)\widetilde{\psi}(q, x^0), \tag{144}$$

where locally

$$\widetilde{\psi}(q, x^0) = \tilde{\rho}^{1/2}(q, x^0)\exp\left(\frac{\mathrm{i}}{f}\tilde{\lambda}(q, x^0)\right).$$

The general solution to (144) is a linear superposition of the vacuum states. Due to (133) and (141), we have for this solution, $P_m^Q(q, x) = 0$ $(m = 1, 2, 3)$, while the random energy density is, in general, time dependent. On the other hand, its expectation value is a constant. In fact, due to (143'), (133) gives

$$\varepsilon^Q(q, x^0) = -\frac{\partial}{\partial x^0}\tilde{\lambda}(q, x^0) = \mathrm{i}\frac{f}{2}\frac{1}{\tilde{\psi}\tilde{\psi}^*}\left(\tilde{\psi}^*\frac{\partial}{\partial x^0}\tilde{\psi} - \tilde{\psi}\frac{\partial}{\partial x^0}\tilde{\psi}^*\right), \tag{145}$$

so that

$$\int \mathrm{d}q\,\varepsilon^Q(q, x^0)\tilde{\rho}(q, x^0) = \bar{w} = \int \mathrm{d}q\,\tilde{\psi}^*(q, x^0)\,\mathrm{i}f\frac{\partial}{\partial x^0}\tilde{\psi}^*(q, x^0). \tag{146}$$

Now we discuss the very interesting case of time-independent solutions, by considering the situation

$$\frac{\partial\lambda^0}{\partial q}(q, x) = 0, \tag{147}$$

while, in general, $\frac{\partial}{\partial q}\lambda^m(q,x) \neq 0$. Also in this case $P_m^Q(q,x) = 0$, but now there appear important features of the random energy density.

As a consequence of (147) we have $\rho(q,x) = \hat{\rho}(q,\vec{x})$, $\lambda^m(q,x) = \hat{\lambda}^m(q,\vec{x})$, $(\vec{x} = (x^1, x^2, x^3), m = 1, 2, 3)$ while, taking into account (128), we can write $\frac{\partial}{\partial x^0}\lambda^0(q,x) = g(\vec{x})$. By introducing $h^m(\vec{x})$, such that $\frac{\partial}{\partial x^m}h^m(\vec{x}) = g(\vec{x})$, we can absorb $h^m(\vec{x})$ within $\hat{\lambda}^m(q,\vec{x})$, by considering $\hat{\lambda}^m(q,\vec{x}) + h^m(\vec{x})$, which we recall $\hat{\lambda}^m(q,\vec{x})$. Our system (125) and (128) becomes

$$\frac{\partial}{\partial x^m}\hat{\rho}(q,\vec{x}) + \frac{1}{\eta}\frac{\partial}{\partial q}\left(\hat{\rho}(q,\vec{x})\frac{\partial}{\partial q}\hat{\lambda}_m(q,\vec{x})\right) = 0 \qquad (m = 1, 2, 3), \tag{148}$$

$$\frac{\partial}{\partial x^m}\hat{\lambda}^m(q,\vec{x}) - \frac{1}{2\eta}\sum_{m=1}^{3}\left(\frac{\partial}{\partial q}\hat{\lambda}_m(q,\vec{x})\right)^2 + V(q) - \frac{f^2}{2\eta\hat{\rho}^{1/2}}\frac{\partial^2}{\partial q^2}\hat{\rho}^{1/2}(q,\hat{x}) = 0. \tag{149}$$

In order to see some basic aspect of this system, we fix the attention on its possible spherical symmetric solutions. We start from the ansatz

$$\hat{\lambda}^m(q,\vec{x}) = -\hat{\lambda}^m(q,r)\frac{x^m}{r} \qquad (r = |\vec{x}|, m = 1, 2, 3). \tag{150}$$

We infer then from (148) $\hat{\rho}(q,\vec{x}) = \hat{\rho}(q,r)$. The system (148) and (149) becomes

$$\frac{\partial}{\partial r}\hat{\rho}(q,r) + \frac{1}{\eta}\frac{\partial}{\partial q}\left(\hat{\rho}(q,r)\frac{\partial\hat{\lambda}(q,r)}{\partial q}\right) = 0, \tag{148$'$}$$

$$-\frac{\partial}{\partial r}\hat{\lambda}(q,r) - \frac{2}{r}\hat{\lambda}(q,r) - \frac{1}{2\eta}\left(\frac{\partial}{\partial q}\hat{\lambda}(q,r)\right)^2 + V(q) - \frac{f^2}{2\eta\hat{\rho}^{1/2}(q,r)}\frac{\partial^2}{\partial q^2}\hat{\rho}^{1/2}(q,r) = 0. \tag{149$'$}$$

This cannot be translated into a Schrödinger type equation, due to the minus sign in the third term of (149$'$). However it is useful to introduce a transformation involving two real, 'conjugate', functions, as is done in diffusion theory [21]. We consider two real functions, $\phi(q,r)$ and $\tilde{\phi}(q,r)$, such that

$$\rho(q,r) = \tilde{\phi}(q,r)\phi(q,r), \tag{151}$$

$$\hat{\lambda}(q,r) = \frac{1}{3}w_0 r - \frac{f}{2}\log\frac{\phi(q,r)}{\tilde{\phi}(q,r)} \qquad (\text{when } \rho > 0). \tag{152}$$

In terms of $\phi(q,r)$ and $\tilde{\phi}(q,r)$, the system (148$'$) and (149$'$) becomes

$$-f\frac{\partial}{\partial r}\phi(q,r) = \left(-\frac{f^2}{2\eta}\frac{\partial^2}{\partial q^2} + V(q) - w_0 + \frac{f}{r}\log\frac{\phi(q,r)}{\tilde{\phi}(q,r)}\right)\phi(q,r), \tag{153}$$

$$f\frac{\partial}{\partial r}\tilde{\phi}(q,r) = \left(-\frac{f^2}{2\eta}\frac{\partial^2}{\partial q^2} + V(q) - w_0 + \frac{f}{r}\log\frac{\phi(q,r)}{\tilde{\phi}(q,r)}\right)\tilde{\phi}(q,r). \tag{154}$$

We are interested to bounded solutions of this system.

We note that, if we neglect the term $(f/r)\log\left(\phi(q,r)/\tilde{\phi}(q,r)\right)$, the resulting equations

$$-f\frac{\partial}{\partial r}\phi_0(q,r) = \left(-\frac{f^2}{2\eta}\frac{\partial^2}{\partial q^2} + V(q) - w_0\right)\phi_0(q,r), \tag{155}$$

$$f\frac{\partial}{\partial r}\widetilde{\phi}_0(q,r) = \left(-\frac{f^2}{2\eta}\frac{\partial^2}{\partial q^2} + V(q) - w_0\right)\widetilde{\phi}_0(q,r) \tag{156}$$

can be easily solved. Equation (155), which is a diffusion equation, admits non-negative solutions, bounded for every r, having the structure

$$\phi_0(q, r) = c_0 \psi_0(q) + c_1 \psi_1(q) \, \mathrm{e}^{-\frac{(w_1 - w_0)}{f} r} + \cdots + c_n \psi_n(q) \mathrm{e}^{-\frac{(w_n - w_0)}{f} r} + \cdots. \tag{157}$$

On the other hand, (156) admits only one solution, which is bounded for large r, that is

$$\widetilde{\phi}_0(q, r) = \widetilde{c}_0 \psi_0(q). \tag{158}$$

Now we take $c_0 = \widetilde{c}_0 (\neq 0)$, so that, for large r,

$$\frac{f}{r} \log \frac{\phi_0(q, r)}{\widetilde{\phi}_0(q, r)} = f \frac{c_1}{c_0} \frac{1}{r} \frac{\psi_1(q)}{\psi_0(q)} \mathrm{e}^{-\frac{(w_1 - w_0)}{f} r} + O\left(\frac{\mathrm{e}^{-\frac{(w_2 - w_0)}{f} r}}{r} \right) \tag{159}$$

(assuming $c_1 \neq 0$). Due to (159), we can apply to the system (153) and (154), for large r, the method of successive approximations, starting from $\phi_0(q, r)$ and $\widetilde{\phi}_0(q, r)$. As a first improvement we obtain

$$\begin{aligned} \phi(q, r) &\simeq \phi_0(q, r) + \int_r^{+\infty} \frac{\phi_0(q, r')}{r'} \log \frac{\phi_0(q, r')}{\widetilde{\phi}_0(q, r')} \, \mathrm{d}r' \\ &= c_0 \psi_0(q) + c_1 \psi_1(q) \left(1 + \frac{f}{w_1 - w_0} \cdot \frac{1}{r} - \frac{f^2}{(w_1 - w_0)^2} \frac{1}{r^2} + \cdots \right) \mathrm{e}^{-\frac{(w_1 - w_0)}{f} r} + \cdots, \end{aligned} \tag{160}$$

$$\widetilde{\phi}(q, r) \simeq c_0 \psi_0(q) - c_1 \psi_1(q) \left(\frac{f}{w_1 - w_0} \cdot \frac{1}{r} - \frac{f^2}{(w_1 - w_0)^2} \frac{1}{r^2} + \cdots \right) \mathrm{e}^{-\frac{(w_1 - w_0)}{f} r} + \cdots. \tag{161}$$

The last dots represent terms of order $\exp(-2(w_1 - w_0)r/f)$ or $\exp(-(w_2 - w_0)/f)$, or smaller than these. From (151), (160) and (161) we obtain that, for large r,

$$\rho(q, r) \simeq \psi_0^2(q) + c_1 \psi_0(q) \psi_1(q) \mathrm{e}^{-\frac{(w_1 - w_0)}{f} r} + \cdots, \tag{162}$$

where we have taken $c_0 = +1$, for a proper normalization. Since $\psi_0(q)$ is the state of the fundamental vacuum, (162) describes a situation in which the physical system of our field is static and confined within a region of the space, centered at the origin, of radius $\sim f/(w_1 - w_0)$. In such a situation we have that, besides the constant vacuum energy density, there is a random energy density confined within a sphere of radius $\sim f/(w_1 - w_0)$.

We obtain analogous results if $c_1 = 0$ and c_2, for example, different from zero. We note that, due to the invariance of our equations as regards spatial translations, the previous considerations on static solutions can be applied to a generic sphere with its center at an arbitrary fixed point of the space. Then one can also consider several spheres, with centers having distances larger than $2f/(w_1 - w_0)$. The above results seem to have some appealing aspects. Their interpretation and a further analysis of our equations for specific models will be given elsewhere. We note that in the case of mechanical systems, where we have only the parameter t, the balance between two competitive diffusion processes makes the H-theorem ineffective. On the other hand, as we have seen, in the case of fields this balance does not prevent that a H-theorem be operative in the spacelike directions.

We add a brief comment on the dynamical behavior of our system (125) and (128). We can also introduce for fields a quantity like the wavefunction of the mechanical systems. However, in this case, the notion of wavefunction is associated with some covering of the spacetime through a family of spacelike surfaces. Each such a covering determines a particular wavefunction. To fix the ideas, let us consider the standard family of spacelike surfaces, each

having a constant 'time' x_0. To this family we can associate the wavefunction $\psi(q, x^0|\vec{x})$, parametrized by $\vec{x}$, given locally by

$$\psi(q, x^0|\vec{x}) = \rho^{1/2}(q, x^0|\vec{x}) \exp\left(\frac{i}{f}\lambda^0(q, x^0|\vec{x})\right).$$

Due to section 3, the system (125) and (128) can be translated in the system

$$i f \frac{\partial}{\partial x^0}\psi(q, x^0|\vec{x}) = -\frac{f^2}{2}\partial_i(\eta^{ij}(q)\partial_j\psi(q, x^0|\vec{x}))$$
$$+ \left(\frac{\partial}{\partial x^m}\lambda^m(q, x^0|\vec{x}) + \frac{1}{2}\eta^{ij}(q)\partial_i\lambda_m(q, x^0|\vec{x})\partial_j\lambda^m(q, x^0|\vec{x})\right.$$
$$\left. + V(q)\right)\psi(q, x^0|\vec{x}) = 0, \tag{163}$$

$$\frac{\partial}{\partial x^m}(\bar{\psi}\psi) + \partial_k(\bar{\psi}\psi\eta^{kj}\partial_j\lambda_m) = 0 \qquad (m = 1, 2, 3). \tag{164}$$

We have then for $\psi(q, x^0|\vec{x})$ a Schrödinger equation, in which, besides the known function $V(q)$, there is also a 'self'-potential, constrained by equations (164). As a result, even if we introduce the above wavefunction, the linearity found for the mechanical system is lost in the case of the fields.

However, it is possible that linearity requires a mathematical object more general than $\psi(q, x^0|\vec{x})$.

We conclude here with a few remarks. The previous approach for scalar fields is based on a particular well-defined probabilistic scheme, which makes use of ordinary functions. On the other hand, the standard approach of the quantum theory of fields is based on a more general probabilistic scheme which is of functional type, but ill-defined. However the probabilistic distributions of our approach could be considered or imposed, as the appropriate marginal distributions of functional distributions which are not well defined.

Our approach requires some further development in order to treat the case of singular Lagrangians. An analysis of the Dirac fields is particularly important in order to see how the previous probabilistic scheme for scalar fields must be modified. A further important problem is to investigate the application of our approach to the case of the gravitational field.

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